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On Quasi-orthogonal Polynomials

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Chihara [On quasi orthogonal polynomials, *Proc. Amer. Math. Soc.* **8** (1957), 765–767] has shown that quasi-orthogonal polynomials satisfy a three-term recurrence relation with polynomial coefficients. In this paper it is shown that, if a sequence of polynomial coefficients is given with some particular properties, then there exists a unique sequence of monic polynomials $(\{U_n\}_{n\in\mathbb{N}}$ and $U_0=1\}$ which satisfies a three-term recurrence relation whose polynomial coefficients are those given. The polynomials are quasi-orthogonal of order 1 with respect to a unique linear functional of moments. Some new properties of the quasi-orthogonal polynomials of order 1 are also proved. © 1990 Academic Press, Inc.

1. QUASI-ORTHOGONAL POLYNOMIALS

The quasi-orthogonal polynomials of order 1 introduced by Riesz [8] were generalized for any order by Chihara [2]. In particular Chihara proved that these polynomials satisfy a three-term recurrence relation with polynomial coefficients.

Some new properties have been given by Dickinson [4] and Brezinski [1] for quasi-orthogonal polynomials of order 1.

More recently Maroni [7], using the quasi-orthogonal polynomials of order $q-1$ to obtain a characterisation of the semi-classical polynomials, gives some properties linking the sets of orthogonal polynomials and those of quasi-orthogonal polynomials.

Finally new properties of the quasi-orthogonal polynomials of order $q-1$ are given by Draux [6], when the linear functional is semi-definite, that is when some Hankel determinants can be zero.

Let c be a linear functional acting on the vector space P of polynomials with complex coefficients. The moments of this functional are given by

$$c(x^i)=c_i, \qquad \forall i\in\mathbb{N}.$$

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The functional c will be said definite if all the Hankel matrices,

$$M_k = (c_{i+j})_{i,j=0}^{k-1}, \qquad \forall k \in \mathbb{N},\ k \geqslant 1$$

have an inverse.

In all the sequel of this first section, c will be assumed to be definite. In this case there exists a unique sequence of monic orthogonal polynomials with respect to c which satisfy a three-term recurrence relation,

$$P_{k+1} = (x + B_{k+1})\, P_k + C_{k+1} P_{k-1}$$

with $C_{k+1} \neq 0$ and the initializations: $P_{-1} = 0$ and $P_0 = 1$. (See for instance Brezinski [1].)

A second sequence $\{Q_i\}_{i \in \mathbb{N}}$ of polynomials satisfies the same recurrence relation with the initializations Q_{-1} equal to an arbitrary non zero constant c_0 and $Q_0 = 0$. In this case, $C_1 = 1$. These polynomials Q_i are the second kind orthogonal polynomials or the associated polynomials of P_i with respect to the linear functional c.

Throughout this paper, q will denote a positive integer.

DEFINITION 1 [7]. A sequence of polynomials $\{U_n\}_{n \in \mathbb{N}}$, such that the degree of U_n is equal to n for any n belonging to $\mathbb{N}$,

(i) is said quasi-orthogonal of order $q-1$ if

$\forall k \in \mathbb{N}$ such that $k \geqslant q-1$, $c(x^l U_k) = 0$, $\forall l \in \mathbb{N}$ such that $0 \leqslant l \leqslant k-q$, and $\exists r \in \mathbb{N}$, $r \geqslant q-1$ such that $c(x^{r-q+1} U_r) \neq 0$.

(ii) is said strictly quasi-orthogonal of order $q-1$ if

$\forall k \in \mathbb{N}$ such that $k \geqslant q-1$, $c(x^l U_k) = 0$, $\forall l \in \mathbb{N}$ such that $0 \leqslant l \leqslant k-q$, and $c(x^{k-q+1} U_k) \neq 0$.

In this paper we are only interested in the monic quasi-orthogonal polynomials of order 1. Each quasi-orthogonal polynomial can be expressed as follows (see Chihara [2]):

$$U_k = P_k + a_k P_{k-1}, \qquad \forall k \in \mathbb{N}. \tag{1}$$

a_k is an arbitrary complex constant.

Obviously, U_k is strictly quasi-orthogonal of order 1 if and only if $a_k \neq 0$.

Chihara [2] has shown that the quasi-orthogonal polynomials satisfy a three-term recurrence relations. To obtain this relation in the case of quasi-orthogonal polynomials of order 1, we compute the first unknown U_k of the following linear system, $\forall k \in \mathbb{N}$ such that $k \geqslant 3$,

$$\begin{bmatrix} 0 & a_{k-2} & 1 & 0 & 0 \\ 0 & 0 & a_{k-1} & 1 & 0 \\ -1 & 0 & 0 & a_k & 1 \\ 0 & C_{k-1} & x+B_{k-1} & -1 & 0 \\ 0 & 0 & C_k & x+B_k & -1 \end{bmatrix} \begin{bmatrix} U_k \\ P_{k-3} \\ P_{k-2} \\ P_{k-1} \\ P_k \end{bmatrix} = \begin{bmatrix} U_{k-2} \\ U_{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, this relation is

$$E_k(x)\,U_k(x) = F_k(x)\,U_{k-1}(x) + G_k(x)\,U_{k-2}(x), \qquad \forall k \in \mathbb{N} \text{ such that } k \geqslant 3, \tag{2}$$

where $-E_k$ is equal to the determinant of the linear system, $-F_k$ is the second cofactor and $-G_k$ the first cofactor of this determinant.

The three polynomials E_k, F_k, and G_k are

$$\begin{aligned} E_k(x) &= a_{k-2}(x+B_{k-1}+a_{k-1}) - C_{k-1} \\ F_k(x) &= (a_{k-2}(x+B_{k-1}) - C_{k-1})(x+B_k+a_k) + a_{k-2}C_k, \\ G_k(x) &= C_{k-1}(a_{k-1}(x+B_k+a_k) - C_k). \end{aligned} \tag{3}$$

The first relations are

$$\begin{aligned} U_1 &= F_1 U_0 && \text{with} \quad \deg F_1 = 1, \\ E_2 U_2 &= F_2 U_1 + G_2 U_0 && \text{with} \quad E_2 = 1,\ F_2 = x + B_2 + a_2, \\ & && \text{and } G_2 = -(a_1(x+B_2+a_2) - C_2). \end{aligned} \tag{4}$$

The following property is obvious and has been extended to the case of quasi-orthogonal polynomial of order $q-1$ (see Draux [6]).

Property 2. Say

> $\forall k \in \mathbb{N}$ such that $k \geqslant 4$, $G_{k-1}(x) = C_{k-2}E_k(x)$. For $k=2$ and 3, $E_k = G_{k-1}$.

Another property will be used in the sequel when $a_{k-1}a_{k-2} \neq 0$, that is to say when the degrees of E_k and G_k are equal to 1.

Property 3. If $a_{k-1}a_{k-2} \neq 0$ and if x is a common zero of two of the polynomials E_k, F_k, and G_k, then x is also a zero of the third one.

Proof. F_k can be written

$$F_k = \frac{E_k G_k}{C_{k-1} a_{k-1}} - \frac{a_{k-2}}{C_{k-1}} G_k + \frac{C_k}{a_{k-1}} E_k, \tag{5}$$

and the property is immediately verified. ▮

2. Polynomials Satisfying a Three-Term Recurrence Relation

In this part, our aim is to find the sequences of polynomials satisfying a three-term recurrence relation of type (2) and to prove that these polynomials are quasi-orthogonal of order 1 with respect to a unique linear functional of moments when the first moment c_0 is fixed.

A sequence $\{G_i\}_{i \geq 1}$ of non identically zero polynomials and a sequence $\{C_i\}_{i \geq 1}$ of complex numbers are assumed to be known with the following assumptions:

(i) $G_1 = 1$,

(ii) the degree of G_k is equal to 0 or 1,

(iii) at least one of G_k's polynomials has a degree equal to 1,

(iv) $C_i \neq 0$, $\forall i \geq 1$; $C_1 = 1$,

(v) If the degree of G_{k+1} is zero, then $G_{k+1} = -C_k C_{k-1}$. Let numbers a_k $(k \geq 0)$ be defined by writing the coefficient of x in $G_{k+1}(x)$ as $a_k C_k$. (In particular, because of (i), $a_0 = 0$).

From the sequences $\{G_i\}_{i \geq 1}$ and $\{C_i\}_{i \geq 1}$ two other sequences $\{E_i\}_{i \geq 2}$ and $\{F_i\}_{i \geq 1}$ of polynomials are deduced thanks to the following assumptions:

(vi) $E_k = G_{k-1}$ for $k = 2$; $C_{k-2} E_k = G_{k-1}$, $\forall k \in \mathbb{N}$ such that $k \geq 3$,

(vii) F_1 is equal to an arbitrary monic polynomial of degree 1 exactly, and $\forall k \geq 2$, if $a_{k-1} \neq 0$ then

$$F_k(x) = \frac{E_k(x)\, G_k(x)}{C_{k-1} a_{k-1}} - \frac{a_{k-2}}{C_{k-1}} G_k(x) + \frac{C_k}{a_{k-1}} E_k(x) \tag{6}$$

and if $a_{k-1} = 0$ then

$$F_k(x) = E_k(x)\, L_k(x) + C_k a_{k-2}, \tag{7}$$

where $L_k(x)$ is a polynomial of degree 1 exactly.

Remark. Property 3 holds.

With the three sequences $\{E_i\}_{i\geqslant 2}$, $\{F_i\}_{i\geqslant 1}$, and $\{G_i\}_{i\geqslant 1}$ a fourth sequence $\{U_i\}_{i\in\mathbb{N}}$ of polynomials will be generated by

$$E_k U_k = F_k U_{k-1} + G_k U_{k-2}, \qquad \forall k \geqslant 2 \tag{8}$$

with the initializations: $U_0 = 1$ and $U_1 = F_1 U_0$.

From the assumptions v, vi, and vii it is easy to see that $\deg F_k = 1 + \deg E_k$ and the leading coefficients of the two polynomials are equal. Then:

Property 4. If the sequence $\{U_i\}_{i\in\mathbb{N}}$ exists, U_i is monic and $\deg U_i = i$, $\forall i \in \mathbb{N}$.

THEOREM 5. *The sequence* $\{U_i\}_{i\in\mathbb{N}}$ *exists.*

Proof. The subsequence $\{U_i\}_{i=0}^{k-1}$ is assumed to be already computed.

If $\deg E_k = 0$, U_k exists.

If $\deg E_k = 1$ the relation (8) shows that $E_{k-1}U_{k-1} - F_{k-1}U_{k-2}$ is divisible by G_{k-1}, that is to say by E_k. Thanks to relation (6) this last quantity is equal to

$$E_{k-1}U_{k-1} - \frac{E_{k-1}G_{k-1}}{C_{k-2}a_{k-2}}U_{k-2} + \frac{a_{k-3}}{C_{k-2}}G_{k-1}U_{k-2} - \frac{C_{k-1}}{a_{k-2}}E_{k-1}U_{k-2},$$

for $a_{k-2} \neq 0$.

Therefore $E_{k-1}(U_{k-1} - (C_{k-1}/a_{k-2})U_{k-2})$ is divisible by G_{k-1}.

Let us set $j = k-1$.

(I) (i) If $\deg E_j = 0$, $U_j - (C_j/a_{j-1})U_{j-1}$ is divisible by G_j. Then goto (II).

(ii) If $\deg E_j = 1$ and if E_j is not divisible by G_j, then $U_j - (C_j/a_{j-1})U_{j-1}$ is divisible by G_j. Then goto (II).

(iii) If $\deg E_j = 1$ and if E_j is divisible by G_j, then F_j also is divisible by G_j (Property 3).

Then the quantity $U_j - (C_j/a_{j-1})U_{j-1}$ is equal to

$$\frac{G_j}{C_{j-1}a_{j-1}}U_{j-1} - a_{j-1}\left(U_{j-1} - \frac{C_{j-1}}{a_{j-2}}U_{j-2}\right), \tag{9}$$

by using the relations (6) and (8), and the assumption (vi).

But thanks to relations (8) it also is obvious that

$$E_{j-1}(U_{j-1} - (C_{j-1}/a_{j-2})U_{j-2}) \text{ is divisible by } G_{j-1}\text{, thus by } G_j. \tag{10}$$

Then replace j by $j-1$ and goto (I).

(II) Remark that if $\deg E_j = 1$ and if E_j is divisible by G_j, $\forall j \in \mathbb{N}$ $3 \leqslant j \leqslant k-1$, the last iteration will be (I)(i) because $E_2 = 1$.

The result is

$$U_j - \frac{C_j}{a_{j-1}} U_{j-1} \text{ is divisible by } G_j.$$

But thanks to the relations (9) and (10) it is obvious that

$$U_{k-1} - \frac{C_{k-1}}{a_{k-1}} U_{k-2} \text{ is divisible by } G_{k-1}. \tag{11}$$

Now to compute U_k, $F_k U_{k-1} + G_k U_{k-2}$ must be divisible by E_k, that is to say by G_{k-1}.

If $a_{k-1} \neq 0$ and thanks to relation (6) it is easy to see that

$$G_k \frac{a_{k-2}}{C_{k-1}} \left(U_{k-1} - \frac{C_{k-1}}{a_{k-2}} U_{k-2} \right) \text{ must be divisible by } G_{k-1},$$

which is also verified (see (11)).

If $a_{k-1} = 0$ and thanks to relation (7) we have the following result:

$$C_k a_{k-2} \left(U_{k-1} - \frac{C_{k-1}}{a_{k-2}} U_{k-2} \right) \text{ must be divisible by } G_{k-1},$$

which is also verified (see (11)). ∎

Moreover the following property is given by the proof of the preceding theorem:

THEOREM 6. *If* $\deg E_k = 1$, $U_{k-1} - (C_{k-1}/a_{k-2})\, U_{k-2}$ *is divisible by* E_k.

A new sequence $\{P_j\}_{j \in \mathbb{N}}$ of polynomials will be generated by

$$E_{j+2} P_j = a_j U_{j+1} - C_{j+1} U_j. \tag{12}$$

These new polynomials verify the following:

THEOREM 7. $\forall i \in \mathbb{N}$,

(i) $\deg P_i = i$,

(ii) P_i *is monic*,

(iii) $U_{i+1} = P_{i+1} + a_{i+1} P_i$. (13)

Proof. The properties (i) and (ii) are obvious thanks to relation (12).
(iii) If $a_{i+1} = 0$, then $U_{i+1} = P_{i+1}$ (given by relation (12)).

If $a_{i+1} \neq 0$, the two members of relation (12) (with $j=i+1$) are multiplied by E_{i+2}, and the quantity $E_{i+2}U_{i+2}$ is replaced by its expression given by relation (8). Finally $a_{i+1}F_{i+2}$ is replaced by its expression given by relation (6). Then the following relation is obtained:

$$E_{i+2}E_{i+3}P_{i+1} = E_{i+2}E_{i+3}U_{i+1} - a_{i+1}E_{i+3}(a_i U_{i+1} - C_{i+1}U_i).$$

But $a_i U_{i+1} - C_{i+1}U_i = E_{i+2}P_i$ and after simplification by $E_{i+2}E_{i+3}$, the relation (13) is obtained.

It is obvious that relation (8) has two independent solutions. The first solution sequence is obtained from the initializations: $U_0 = 1$ and $U_1 = F_1$. The second solution sequence will be given from the independent initializations: $V_0 = 0$ and $V_1 =$ arbitrary non zero constant c_0.

A similar proof as that of Theorem 5 shows that the second sequence $\{V_i\}_{i\in\mathbb{N}}$ also exists, and the following theorem obviously holds:

THEOREM 8. $\forall i \in \mathbb{N}$ *such that* $i \geqslant 1$

(i) $\deg V_i = i - 1$,

(ii) *the leading coefficient of* V_i *is equal to* c_0,

(iii) *If* $\deg E_{i+2} = 1$, $V_{i+1} - (C_{i+1}/a_i)\, V_i$ *is divisible by* E_{i+2}.

Another new sequence $\{Q_j\}_{j\in\mathbb{N}}$ of polynomials can be generated by

$$E_{j+2}Q_j = a_j V_{j+1} - C_{j+1}V_j. \tag{14}$$

A result similar to Theorem 7 can be proved:

THEOREM 9. $\forall i \in \mathbb{N}$, *such that* $i \geqslant 1$,

(i) $\deg Q_i = i - 1$,

(ii) *the leading coefficient of* Q_i *is equal to* c_0,

(iii) $V_i = Q_i + a_i Q_{i-1}$. (15)

From relation (8) which is satisfied by the polynomials U_i and V_i it is easy to obtain:

$$E_i(U_i V_{i-1} - U_{i-1}V_i) = -G_i(U_{i-1}V_{i-2} - U_{i-2}V_{i-1}).$$

All the relations, $\forall i \in \mathbb{N}$ such that $2 \leqslant i \leqslant k$ are multiplied by each other and after simplification the following result is obtained:

THEOREM 10. $\forall k \geqslant 2$,

$$U_k V_{k-1} - U_{k-1}V_k = (-1)^k c_0 \prod_{j=2}^{k-2} C_j G_k. \tag{16}$$

If $k \leqslant 3$ *the product is taken equal to* 1.

COROLLARY 11. (i) *At most one of the pairs* (U_k, U_{k-1}) *and* (V_k, V_{k-1}) *can have a common zero which is a zero of* G_k.

(ii) *The pair* (U_k, V_k) *never has a common zero,* $\forall k \in \mathbb{N}$.

Proof. (i) Is a direct consequence of the relation (16).

(ii) If, for instance, U_k and V_k had a common zero, it would be a zero of G_k, but the relations (12) and (14) with $j = k-1$ show that it would also be a zero of U_{k-1} and V_{k-1}, and a contradiction would be obtained with the first property of this corollary. ∎

From the sequences $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ some monic polynomials $\bar{P}_i$ and some polynomials $\bar{Q}_i$ whose leading coefficient is c_0, will be given by the following processes:

$$U_0 = 1$$

$$U_1 = (x + \hat{B}_1)\, U_0$$

$$U_2 = (x + \hat{B}_2)\, U_1 + \hat{C}_2 U_0$$

$$U_k = (x + \hat{B}_k)\, U_{k-1} + \hat{C}_k U_{k-2} + \hat{D}_k \bar{P}_{k-3}, \qquad \forall k \geq 3$$

with

$$\deg \bar{P}_{k-3} \leq k-3. \tag{17}$$

$$V_0 = 0$$

$$V_1 = c_0$$

$$V_2 = (x + B_2^*)\, V_1 + C_2^* V_0$$

$$V_k = (x + B_k^*)\, V_{k-1} + C_k^* V_{k-2} + D_k^* \bar{Q}_{k-3}, \qquad \forall k \geq 3$$

with

$$\deg \bar{Q}_{k-3} \leq k-4 \qquad (\text{obviously } \bar{Q}_0 = 0). \tag{18}$$

Relation (17) has already been given by Dickinson [4] by using a property of the quasi-orthogonal polynomials of order 1.

A first result can be proved about the coefficient of the relations (17) and (18).

THEOREM 12. $\hat{B}_k = B_k^*,\ \hat{C}_k = C_k^*,\ \hat{D}_k = D_k^*,\ \forall k \geq 3.$

Proof. Let us multiply relation (17) by V_{k-1} and relation (18) by U_{k-1}. The difference of these two new expressions gives

$$\begin{aligned} U_k V_{k-1} - U_{k-1} V_k &= (\hat{B}_k - B_k^*)\, U_{k-1} V_{k-1} + \hat{C}_k (U_{k-2} V_{k-1} - V_{k-2} U_{k-1}) \\ &\quad + (\hat{C}_k - \hat{C}_k^*)\, V_{k-2} U_{k-1} + \hat{D}_k \bar{P}_{k-3} V_{k-1} - D_k^* \bar{Q}_{k-3} U_{k-1}. \end{aligned} \tag{19}$$

The result is obtained thanks to relation (16) and a comparison of the degrees. ▌

By using relation (16), the following result is obvious:

COROLLARY 13. $\forall k \geq 3$,

(i) *If* $\hat{D}_k \neq 0$, *then* $\deg \bar{P}_{k-3} = 1 + \deg \bar{Q}_{k-3}$

(ii) $\hat{D}_k(\bar{P}_{k-3} V_{k-1} - \bar{Q}_{k-3} U_{k-1})$

$$= (-1)^k c_0 \prod_{j=2}^{k-3} C_j (C_{k-2} G_k - \hat{C}_k G_{k-1}). \tag{20}$$

Two other important theorems can be given about the coefficients and the polynomials $\bar{P}_{k-3}$ and $\bar{Q}_{k-3}$ of relations (17) and (18).

THEOREM 14. *The three following properties are equivalent for* $k \geq 3$:

(i) $a_{k-1} = a_{k-2} = 0$ *or*
$a_{k-1} a_{k-2} \neq 0$ *and the three polynomials* E_k, F_k, *and* G_k *have a common zero.*

(ii) $\hat{D}_k = 0$.

(iii) $D_k^* = 0$.

In this case,

if $a_{k-1} = a_{k-2} = 0$ *then* $\hat{C}_k = C_k$,

if $a_{k-1} a_{k-2} \neq 0$ *then* $\hat{C}_k = \dfrac{a_{k-1}}{a_{k-2}} C_{k-1}$.

Proof. (i) $\Rightarrow$ (ii) and (iii). If $a_{k-1} a_{k-2} \neq 0$ and the three polynomials E_k, F_k, and G_k have a common zero, the relation (8) can be written,

$$U_k = (x + \tilde{B}_k) U_{k-1} + \tilde{C}_k U_{k-2},$$

after having divided by E_k.

It is the same result if $a_{k-1} = a_{k-2} = 0$.

It is obvious in the two cases, these two relations give $\hat{D}_k = 0$.

The same proof also is valid for D_k^*.

(iii) $\Rightarrow$ (ii). Theorem 12 shows that $\hat{D}_k = D_k^*$.

(ii) $\Rightarrow$ (i). The relation (20) gives

$$C_{k-2} G_k = \hat{C}_k G_{k-1}$$

and thus

$$G_k = \hat{C}_k E_k.$$

Therefore

$$a_{k-1} = a_{k-2} = 0 \qquad \text{and} \qquad \hat{C}_k = C_k,$$

or $a_{k-1} a_{k-2} \neq 0$ and G_k is divisible by E_k. The property (3) proves the result. Moreover:

$$\hat{C}_k = \frac{a_{k-1}}{a_{k-2}} C_{k-1}$$

THEOREM 15. *If* $k \geqslant 3$ *and* $\hat{D}_k \neq 0$, *then*:

(i) $P_{k-3} = \bar{P}_{k-3}$,

(ii) $Q_{k-3} = \bar{Q}_{k-3}$,

(iii) $\hat{D}_k = a_{k-1} C_{k-1} - a_{k-2} \hat{C}_k$ *and if* $a_{k-1} = 0$ *then* $C_k = \hat{C}_k$.

Proof. The relation

$$(F_k - E_k(x + \hat{B}_k))\, U_{k-1} + (G_k - E_k \hat{C}_k)\, U_{k-2} = E_k \hat{D}_k \bar{P}_{k-3} \qquad (21)$$

is deduced from the difference between the relation (17) multiplied by E_k and the relation (8).

The relation (21) is only satisfied if

$$F_k - E_k(x + \hat{B}_k) = \text{const.} \qquad \text{and} \qquad G_k - E_k \hat{C}_k \neq 0.$$

$G_k - E_k \hat{C}_k$ could have a degree equal to 0 or 1. If this degree was 0 with $\hat{D}_k \neq 0$, then the degrees of E_k and G_k would be equal to 1, or the degree of E_k would be equal to 1 and that of G_k equal to 0 and $\hat{C}_k$ equal to 0. Moreover $F_k - E_k(x + \hat{B}_k)$ would be equal to 0. In the first case of the degree values F_k would be divisible by E_k which is impossible. In the second case the relation (21) would show that U_{k-2} would be divisible by E_k and the relation (12) that U_{k-1} would also be divisible by E_k. In the same way V_{k-1} and V_{k-2} would be divisible by E_k, which is a contradiction of the first part of Corollary 11.

Thus $G_k - E_k \hat{C}_k$ has degree 1 and $F_k - E_k(x + \hat{B}_k)$ is equal to a non zero constant.

(a) If $a_{k-2} \neq 0$ and $a_{k-1} = 0$, the relation (7) gives

$$F_k - E_k(x + \hat{B}_k) = C_k a_{k-2}.$$

The coefficient of x^{k-1} is equal to zero in relation (21). Thus

$$\hat{C}_k = C_k.$$

Then, by using relation (12) and after having simplified relation (21) by E_k it becomes

$$U_{k-2} = P_{k-2} - \frac{\hat{D}_k}{C_k} \bar{P}_{k-3}.$$

The relation (13) shows that

$$\bar{P}_{k-3} = P_{k-3} \qquad \text{and} \qquad \hat{D}_k = -C_k a_{k-2}.$$

(b) If $a_{k-1} \neq 0$, the coefficient of x is equal to $a_{k-1} C_{k-1} - a_{k-2} \hat{C}_k$ in $G_k - E_k \hat{C}_k$ (this expression also contains the case where $a_{k-2} = 0$).

(i) if $a_{k-2} = 0$, then, by using relation (7), the expression of U_{k-1} provided by relation (8) written with $k = k-1$ and divided by E_{k-1}, and relation (12), the relation (21) becomes

$$\begin{aligned}(-a_{k-1} C_{k-1} L_{k-1} + G_k - E_k \hat{C}_k) U_{k-2} - a_{k-1} C_{k-1}^2 P_{k-3} \\ = -C_{k-1} \hat{D}_k \bar{P}_{k-3}.\end{aligned}$$

Therefore

$$P_{k-3} = \bar{P}_{k-3} \qquad \text{and} \qquad D_k = a_{k-1} C_{k-1}$$

(ii) if $a_{k-2} \neq 0$, the same method gives the following transformed relation (21):

$$\begin{aligned}&\left((a_{k-2} \hat{C}_k - a_{k-1} C_{k-1}) \left(\frac{G_{k-1}}{C_{k-2} a_{k-2}} + \frac{C_{k-1}}{a_{k-2}} \right) + G_k - E_k \hat{C}_k \right) U_{k-2} \\ &- \frac{G_{k-1}}{C_{k-2}} (a_{k-2} \hat{C}_k - a_{k-1} C_{k-1}) P_{k-3} = E_k \hat{D}_k \bar{P}_{k-3}.\end{aligned}$$

The factor of U_{k-2} is a constant, but if it was non zero, then U_{k-2} would be divisible by E_k. The proof given for the degree of $G_k - E_k \hat{C}_k$ shows that it is not possible. Thus

$$P_{k-3} = \bar{P}_{k-3} \qquad \text{and} \qquad \hat{D}_k = a_{k-1} C_{k-1} - a_{k-2} \hat{C}_k.$$

(c) $Q_{k-3} = \bar{Q}_{k-3}$ would be obtained by a similar proof by using the polynomials V_i. ▌

A first practical method now can be given to compute the P_i's. The polynomials U_k are obtained thanks to the relation (8) with the initializations: $U_0 = 1$ and $U_1 = F_1 U_0$. All the polynomials P_{k-3}, for which $\hat{D}$ is different from 0, are deduced from the relation (17). Then, the other polynomials P_i are given by the relation (13). Indeed the following theorem, which is a direct consequence of the Theorems 7 and 14, holds:

THEOREM 16. *If* $\hat{D}_{l-1} \neq 0$, $\hat{D}_j = 0$, $\forall j \in \mathbb{N}$ *such that* $l \leqslant j \leqslant m$ *and* $D_{m+1} \neq 0$, *then one of the two following properties holds:*

(i) *All the* a_j*'s are non zero* $\forall j \in \mathbb{N}$ *such that* $l-2 \leqslant j \leqslant m-1$ *and the sequence of the polynomials* P_j *for any* j *belonging to* $\mathbb{N}$ *such that* $l-3 \leqslant j \leqslant m-3$ *can be generated by the relation,*

$$U_{j+1} = P_{j+1} + a_{j+1} P_j.$$

(ii) *All the* a_j*'s are zero* $\forall j \in \mathbb{N}$ *such that* $l-2 \leqslant j \leqslant m-1$ *and* $P_j = U_j$. *Moreover* $a_m \neq 0$ *and* $a_{l-3} \neq 0$.

Finally the relation (12) is well determined.

Now, it can be proved that the polynomials P_i satisfy a three-term recurrence relation.

THEOREM 17. *The following three-term recurrence relation is satisfied by the polynomials* $\{P_i\}_{i \in \mathbb{N}}$:

$$P_{k+1} = (x + B_{k+1}) P_k + C_{k+1} P_{k-1}, \qquad \forall k \in \mathbb{N} \tag{22}$$

with the initializations $P_{-1} = 0$ *and* $P_0 = 1$.

This relation is also satisfied by the polynomials $\{Q_i\}_{i \in \mathbb{N}}$, *but with the initializations* $Q_{-1} = c_0$ *and* $Q_0 = 0$.

Proof. The relation

$$a_k P_{k+1} = (E_{k+2} - a_k a_{k+1} + C_{k+1}) P_k + C_{k+1} a_k P_{k-1} \tag{23}$$

is obtained by replacing U_i in relation (12) by its expression given by relation (13).

If $a_k \neq 0$, relation (22) is obtained.

If $a_k = 0$, the relation

$$P_{k+1} = (x + \hat{B}_{k+1} - a_{k+1}) P_k + \hat{C}_{k+1} P_{k-1} + (\hat{C}_{k+1} a_{k-1} + \hat{D}_{k+1}) P_{k-2}$$

is deduced from relation (17) in the same way.

The last coefficient is equal to 0 (see Theorem 14(iii) or 15(iii)), and relation (22) holds.

A similar proof could be used for the polynomials Q_i. ∎

A simpler second method can be given to compute the sequences $\{U_i\}_{i \in \mathbb{N}}$ and $\{P_i\}_{i \in \mathbb{N}}$.

If the two sequences $\{C_i\}_{i \geqslant 1}$ and $\{G_i\}_{i \geqslant 1}$ are known, then the two other sequences $\{a_i\}_{i \geqslant 1}$ and $\{E_i\}_{i \geqslant 2}$ can be deduced from them.

If $a_k \neq 0$, P_{k+1} is computed by using relation (23), U_{k+1} is then determined by relation (13).

If $a_k = 0$, U_{k+1} is computed from relation (8), and P_{k+1} is then obtained from relation (13).

The main theorem now can be proved:

THEOREM 18. *If two sequences* $\{C_i\}_{i \geqslant 1}$ *and* $\{G_i\}_{i \geqslant 1}$ *are given satisfying the assumptions* (i)–(vii), *then there exists a linear functional* c *of moments with respect to which the polynomials* $\{P_i\}_{i \geqslant 1}$ *are orthogonal and the polynomials* $\{U_k\}_{k \geqslant 2}$ *are quasi-orthogonal of order* 1 (*strictly quasi-orthogonal of order* 1 *if* $a_k \neq 0$). *This functional is definite and is uniquely determined once the arbitrary non zero moment* c_0 *is fixed.*

Proof. The sequence $\{P_i\}_{i \in \mathbb{N}}$, satisfying a three-term recurrence relation, is orthogonal with respect to a unique linear functional c whose moments c_i are determined by the relations $c(P_i) = 0$, $\forall i \in \mathbb{N}$ such that $i \geqslant 1$ with a non zero arbitrary fixed moment c_0 (it is the Favard theorem: see Chihara [3]). This functional is definite, for $C_{k+1} \neq 0$, $\forall k \in \mathbb{N}$.

Relation (13) then proves the quasi-orthogonality of order 1, for at least one of the polynomials G_k has a degree equal to 1 and therefore $a_{k-1} \neq 0$. ∎

Remark. The associated polynomial of a polynomial u with respect to a linear functional c is defined by

$$c\left(\frac{u(x) - u(t)}{x - t}\right).$$

Q_k and V_k are the polynomials associated to P_k and U_k, respectively, for P_k and Q_k satisfy the same three-term recurrence relation. Thus Q_k is the second kind orthogonal polynomial which is identical with the associated polynomial of P_k with respect to c.

Moreover P_k and Q_k satisfying the same three-term recurrence relation, the associated polynomial of U_k also satisfies the same three-term recurrence relation (8) as U_k. But it is also satisfied by the sequence $\{V_k\}_{k \in \mathbb{N}}$. Thus this associated polynomial is identical to V_k. V_k will be called a quasi-orthogonal polynomial of second kind.

Remark. Let g be the inverse formal power series of the formal power series f, where

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \qquad \text{and} \qquad g(x) = \sum_{i=0}^{\infty} d_i x^i.$$

Therefore

$$f(x)\, g(x) = 1.$$

A new linear functional $d^{(2)}$ can be defined from the moments d_i, $\forall i \in \mathbb{N}$ such that $i \geqslant 2$ by the relations

$$d^{(2)}(x^i) = d_{i+2}.$$

The sequence of orthogonal polynomials $\{R_j^{(2)}\}_{j \in \mathbb{N}}$ with respect to the functional $d^{(2)}$ can be introduced. Then (see Brezinski [1])

$$Q_k(x) = c_0 R_{k-1}^{(2)}(x).$$

Thus Q_k is orthogonal with respect to $d^{(2)}$, and therefore V_k is quasi-orthogonal of order 1 with respect to $d^{(2)}$.

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